JOURNAL OF ECONOMIC THEORY 13, 47–57 (1976)

A Note on the Role of the Transversality Condition in Signalling Capital Overaccumulation*

TAPAN MITRA

Department of Economics, University of Illinois at Chicago Circle, Chicago, Illinois 60680

AND

MUKUL MAJUMDAR

Department of Economics, Cornell University, Ithaca, New York 14853

Received August 8, 1975

I. INTRODUCTION

One of the main results in [1] asserts the existence of *one* system of competitive prices supporting an efficient program relative to which the transversality condition is satisfied; i.e., the sequence of values of inputs converges to zero. In general, however, there may be more than one system of competitive prices associated with an efficient program and in fact, it is possible that the transversality condition holds for one system of competitive prices while it does not hold for another (see the example below). It is, therefore, natural to look for conditions that guarantee that the transversality condition is satisfied for *all* the competitive price systems supporting an efficient program. The purpose of this note is precisely to present such a set of conditions applicable to a fairly extensive class of "closed" multisector models. Besides the standard assumptions on technology, (like continuity, convexity, constant returns, free disposal, and impossibility of free production), we require that the input vectors of all the activities in the von Neumann facet be strictly positive. Under such conditions, the transversality condition is obtained for *all* competitive

* The need to settle the question studied in this note was emphasized by, among others, Professor D. Cass in his detailed comments on our earlier paper with Professor D. McFadden. Our interest in the role of the transversality condition is surely due to a great extent to his related works. An earlier version of the paper was presented at the M.S.S.B. Conference held at the Dartmouth College Conference Center in June 1975. The present version has benefited from the helpful comments of Professors Cass, Shell, and McKenzie and other participants.

prices associated with an efficient program. Interpreted from a different angle, the main result shows that a competitive program violating the transversality condition must necessarily be inefficient. Thus, we have a simple and easily applicable criterion for testing the efficiency of a competitive program.

While the technique of proof leading to the main result is quite independent of any result derived or discussed in [1], the interested reader is referred to that paper for a detailed discussion of the model and the related results of Cass, Malinvaud, and others in the published literature. It should perhaps be mentioned that in the literature on efficient and optimal growth, the "necessity" of the transversality condition—its role in signalling capital overaccumulation in competitive programs—has long been the subject of much discussion.

II. Notation

For any $x = (x^i)$ in R^m, x is *nonnegative* (written $x \geqslant 0$) if $x^i \geqslant 0$; it is *semipositive* (written $x > 0$) if $x \geqslant 0$ and $x \neq 0$; it is *strictly positive* (written $x \gg 0$) if $x^i > 0$ for all i. The set of *all* nonnegative (respectively, strictly positive) m-vectors is denoted by R_+^m (respectively R_{++}^m). The m-vector ω has 1 in each coordinate. The *norm* of x (written $|x|$) is chosen as $|x| = \sum_{i=1}^m |x^i|$. For any two nonzero vectors x and x', the angular distance $d(x, x')$ is given by

$$d(x, x') = |x/|x| - x'/|x'||. \tag{2.1}$$

For any nonempty set F and any vector x in R^m,

$$d(x, F) = \inf_{x' \in F} d(x, x'). \tag{2.2}$$

A sequence $p = (p_t)$ of m-vectors is *nonzero* if $p_t \neq 0$ for at least one t.

III. The Model

We shall recall only the definitions used in the statement of our result, and the assumptions needed. As usual, $\mathscr{T}$ is the technology, a nonempty set in R_+^{2m}. The pair (x, y) of m-vectors belong to $\mathscr{T}$ if it is possible to transform the input vector x into the output vector y in one period, where m is the number of commodities. The four standard assumption on $\mathscr{T}$ are:

(A.1) *$\mathscr{T}$ is a closed convex cone in* R_+^{2m} (continuity, convexity, constant returns).

(A.2) "$(0, y) \in \mathscr{T}$" *implies* "$y = 0$" (impossibility of free production).

(A.3) "$(x, y) \in \mathscr{T}$" *and* "$x' \geqslant x, 0 \leqslant y' \leqslant y$" *imply* $(x', y') \in \mathscr{T}$ (free disposal).

(A.4) *There is* $(\bar{x}, \bar{y}) \in \mathscr{T}$ *with* $\bar{y} \gg 0$ (producibility).

Our next assumption is related to the nature of *the von Neumann equilibria* associated with $\mathscr{T}$. It is known that (A.1) through (A.4) imply that there is a von Neumann equilibrium, i.e., there exist a semipositive input vector $\hat{x} > 0$, a (finite) positive scalar $\hat{\lambda} > 0$, and a semipositive price vector $\hat{p} > 0$ such that

$$\begin{aligned} (\hat{x}, \hat{\lambda}\hat{x}) \in \mathscr{T}, \hat{p}y \leqslant \hat{\lambda}\hat{p}x \quad & \text{for all } (x, y) \in \mathscr{T}, \\ \hat{\lambda} \geqslant \lambda(x, y) \quad & \text{for all } (x, y) \in \mathscr{T}, \end{aligned} \tag{3.1}$$

where

$$\lambda(x, y) = \max[\lambda : y \geqslant \lambda x; x > 0].$$

Without loss of generality $\hat{\lambda}$ is taken to be equal to 1. Since the von Neumann price vector is by no means unique in general, we consider all such price vectors. Formally, let us define

$$\mathscr{P} = \{p \in R^m, p > 0, | p | = 1, py \leqslant px \text{ for all } (x, y) \in \mathscr{T}\}. \tag{3.2}$$

It is trivial to verify that $\mathscr{P}$ is a closed convex set. Recall that the *von Neumann–McKenzie facet* F^* is simply defined as the set of all activities breaking even at any $\hat{p}$ in the relative interior of $\mathscr{P}$, i.e., one has (see [2, p. 171])

$$F^* = \{(x, y) \in \mathscr{T} : \hat{p}y = \hat{p}x\} \qquad \textit{for any } \hat{p} \textit{ in the relative interior of } \mathscr{P}. \tag{3.3}$$

It is known that F^* is a closed convex cone with vertex at (0, 0). Our next assumption requires that for any activity in F^* other than (0, 0), the input vector must be strictly positive. Formally, we have

(A.5) *for any* $(x, y) \in F^*$, *with* $(x, y) \neq (0, 0)$, *one must have* $x \gg 0$.

In particular, the vector $\hat{x}$ of von Neumann stocks defined in (3.1) must also satisfy $\hat{x} \gg 0$. An important consequence of (A.5) is that

$$\textit{any von Neumann price vector } \hat{p} \textit{ is strictly positive.} \tag{3.4}$$

If $\hat{p}^i = 0$ for any i, the activity $(x, 0)$ in $\mathscr{T}$ with $x^i > 0$ and $x^j = 0$ for $j \neq i$, clearly breaks even at $\hat{p}$. Hence it belongs to F^*, contradicting the strict positivity requirement of (A.5), and we get (3.4).

It should be emphasized that (A.5) is somewhat restrictive. A few remarks relating (A.5) to some well-known conditions in the literature on intertemporal resource allocation will be instructive. First, going back to (3.1), recall that if the technology $\mathscr{T}$ is such that the pair $(\hat{x}, \hat{p})$ of von Neumann stocks and prices satisfies the famous condition of *unique profitability* introduced by Radner [4], i.e., if one has

$$\hat{p}y - \hat{p}x < 0 \qquad \text{for all } (x, y) \in \mathscr{T} \text{ with } x \text{ not proportional to } \hat{x}, \tag{3.5}$$

the facet F^* reduces to a unique ray $(\hat{x}, \hat{x})$. Thus, if the Radner condition is satisfied, (A.5) requires that the unique von Neumann stock vector $\hat{x}$ be strictly positive. The Radner condition and the strict positivity of $\hat{x}$ figure prominently in the final state turnpike literature (see [3, pp. 213–219]). In general, a technology $\mathscr{T}$ satisfying (A.1) through (A.5) will by no means satisfy the Radner condition (see [2, p. 173]).

Second, (A.5) does *not* imply the condition of output substitution discussed in [1]. Let

$$\mathscr{T} = \{(x, y): x \geqslant Az, 0 \leqslant y \leqslant Bz \text{ for } z \geqslant 0\}, \tag{3.6}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}.$$

Note that the von Neumann stock vector

$$\hat{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is unique, and the facet F^* consists only of the ray through $(\hat{x}, \hat{x})$, where $\hat{p} = (1, 1, 1)$ is a von Neumann price vector.[1] The technology, however, does not satisfy the condition of output substitution. This can be verified easily by considering

$$(x, y) = \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right)$$

[1] Note that $\hat{p}y \leqslant \hat{p}Bz = 3z_1 + 3z_2/2$ and $\hat{p}x \geqslant \hat{p}Az = 3z_1 + 2z_2$ for all $(z_1, z_2) \geqslant 0$. Thus, $\hat{p}y \leqslant \hat{p}x$ with equality holding for $z_1 > 0$ and $z_2 = 0$.

and noting that the output of the first commodity cannot be increased by reducing the output of the third commodity a little.[2]

A number of conditions for special classes of generalized Leontief and von Neumann models can be used to guarantee (A.5). The interested reader is referred to the extended discussions in [2, 3].

A feasible *production program* from $\mathbf{x}$ is a sequence $(x, y) = (x_t, y_{t+1})$ such that

$$x_0 = \mathbf{x}, \qquad x_t \leqslant y_t \qquad \text{for all } t \geqslant 1,$$
$$(x_t, y_{t+1}) \in \mathscr{T} \qquad \text{for all } t \geqslant 0. \tag{3.7}$$

The *consumption program* $c = (c_t)$ generated by (x, y) is defined as:

$$c_t = y_t - x_t \ (\geqslant 0) \qquad \text{for all } t \geqslant 1. \tag{3.8}$$

We refer to (x, y, c) as a *feasible program* $\mathbf{x}$, it being understood that (x, y) is a production program and c is the corresponding consumption program. A feasible program (x^*, y^*, c^*) from $\mathbf{x}$ is *efficient* if there is no other feasible program (x, y, c) from $\mathbf{x}$ such that $c_t \geqslant c_t^*$ for all t and $c_t > c_t^*$ for some t. A feasible program (x^*, y^*, c^*) is *competitive* if there is a nonzero sequence (p_t^*) of nonnegative price vectors such that for all $t \geqslant 0$ one has

$$0 = p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geqslant p_{t+1}^* y - p_t^* x \qquad \text{for all } (x, y) \text{ in } \mathscr{T}. \tag{3.9}$$

In other words, the *intertemporal profit maximization* condition (3.9) is satisfied for all t. A competitive program (x^*, y^*, c^*) satisfies the *transversality condition* if $p_t^* x_t^*$ goes to zero as t goes to infinity.

IV. The Necessity of the Transversality Condition

We are now in a position to state and prove the main result. Under (A.1) through (A.5), let (x^*, y^*, c^*) from $\mathbf{x} \gg 0$ be efficient and competitive at prices $\langle p_t^* \rangle$ satisfying (3.9). Then the transversality condition is necessarily satisfied. Thus, the asymptotic behavior of $p_t^* x_t^*$ (the value of inputs at the competitive prices) is intimately related to the question of inefficiency of a competitive program due to capital overaccumulation Recall that if the competitive prices $\langle p_t^* \rangle$ associated with a given feasible

[2] We cannot use the first activity *at all* in achieving this substitution, since reduction of the output of the third commodity does not generate any surplus input of the first commodity, which is essential for using the first activity. On the other hand, using the second activity *alone*, such substitution is impossible due to fixed proportions.

program are all strictly positive the transversality condition is sufficient for efficiency. On the other hand, failure of the transversality condition signifies inefficiency of a given competitive program.

THEOREM 4.1. *Under* (A.1) *through* (A.5), *let* (x^*, y^*, c^*) *be an efficient program from* $\mathbf{x} \gg 0$ *and* $\langle p_t^* \rangle$ *be a nonzero sequence of competitive prices satisfying for all* $t \geqslant 0$

$$0 = p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geqslant p_{t+1}^* y - p_t^* x \qquad \text{for all } (x, y) \in \mathscr{T}. \tag{4.1}$$

It follows that

$$\lim_{t \to \infty} p_t^* x_t^* = 0. \tag{4.2}$$

For a convenient organization of the proof, let us note three preliminary results that provide us with three key steps.

PROPOSITION 4.1. *For any* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $d[(x, y); F^*] \geqslant \epsilon$ *implies*

$$\hat{p}y \leqslant (1 - \delta)\hat{p}x. \tag{4.3}$$

Proof. This, of course, is the famous value-loss lemma of the turnpike literature. This version is in [2, Lemma 4]. Q.E.D.

PROPOSITION 4.2. *There is* $\alpha > 0$ *such that* "$(x, y) \in F^*$, $|(x, y)| = 1$" *implies* $|x| \geqslant \alpha$.

Proof. If not, there is a sequence $(x^n, y^n) \in F^*$ with $|(x^n, y^n)| = 1$ and $\lim_{n \to \infty} |x^n| = 0$. But (y^n) being bounded, one has a subsequence $(x^{n'}, y^{n'})$ in F^*, $|(x^{n'}, y^{n'})| = 1$ converging to (x, y) in F^*, with $x = 0$ and $|y| \neq 0$. This contradicts (A.2). Q.E.D.

PROPOSITION 4.3. *There is* $\mathbf{e}_\alpha = (e_\alpha, \ldots, e_\alpha) = e_\alpha \boldsymbol{\omega} \gg 0$ *such that* "$(x, y) \in F^*$, $|(x, y)| = 1$" *implies* $x \geqslant e_\alpha \boldsymbol{\omega}$.

Proof. Note that the set

$$C_\alpha = \{x \in R_+^m \text{: for some } y \geqslant 0, (x, y) \in F^*, |x| = \alpha\} \tag{4.4}$$

is *a compact subset of* R^m *contained in* R_{++}^m. It is obviously bounded, and by (A.5) is contained in R_{++}^m. To show that it is closed, take a sequence x^n in C_α converging to some $x \geqslant 0$. Clearly $|x| = \alpha$. By definition of C_α, there is a corresponding sequence $y^n \geqslant 0$, such that $(x^n, y^n) \in F^*$. Recall that $|x^n| = \alpha$, implies that (y^n) is bounded (see [4, p. 102].) Hence there is a convergent subsequence $(x^{n'}, y^{n'})$ tending to some

(x', y'). Since F^* is closed, $(x', y') \in F^*$. Since x^n converges to x, any subsequence $x^{n'}$ must converge to x, so that $x = x'$. Thus, $(x, y') \in F^*$, and $|x| = \alpha$ imply that $x \in C_\alpha$, completing the proof of closedness.

As C_α is a compact subset of R^m_{++}, it can be covered by a *finite* number of closed balls each of which is also contained in R^m_{++}. Hence it is obvious that there is some $e_\alpha \boldsymbol{\omega} \gg 0$ such that $x \in C_\alpha$ implies $x \geqslant e_\alpha \boldsymbol{\omega}$. If $(x, y) \in F^*$ and $|(x, y)| = 1$, one has $|x| \geqslant \alpha$ by Proposition 4.2. Note that if for any $(x, y) \in F^*$, $|(x, y)| = 1$ one in fact has $|x| > \alpha$, we can find $\beta > 0$ and $\beta < 1$ such that $|\beta x| = \alpha$. Since F^* is a cone, $(\beta x, \beta y) \in F^*$ so that $\beta x \in C_\alpha$. Hence $\beta x \geqslant e_\alpha \boldsymbol{\omega}$ and $x \geqslant (1/\beta)\, e_\alpha \boldsymbol{\omega} > e_\alpha \boldsymbol{\omega}$. Q.E.D.

Proof of Theorem 4.1. The difficult step in the proof is the assertion

$$\text{Lim inf}\, |x_t^*| = 0. \tag{4.5}$$

Postponing the proof of (4.5) for the moment, let us see how (4.2) follows from (4.5). Since (4.1) is satisfied, one has for any finite T

$$0 \leqslant p_T{}^* x_T{}^* = p_0{}^* \mathbf{x} - \sum_{t=1}^{T} p_t{}^* c_t{}^*. \tag{4.6}$$

Since $S_T = \sum_{t=1}^{T} p_t{}^* c_t{}^*$ is monotonically nondecreasing and bounded above by $p_0{}^*\mathbf{x}$, $\lim_{T\to\infty} \sum_{t=1}^{T} p_t{}^* c_t{}^*$ exists. This in turn implies from (4.6) that $\lim_{T\to\infty} p_T{}^* x_T{}^*$ exists, and of course, $\lim_{T\to\infty} p_T{}^* x_T{}^* \geqslant 0$.

Consider any von Neumann stock vector $\hat{x} \gg 0$. Recall that the von Neumann growth factor in $\mathscr{T}$ is taken to be equal to 1, so that $(\hat{x}, \hat{x}) \in \mathscr{T}$. We use (4.1) to have

$$p_{t+1}^* \hat{x} \leqslant p_t{}^* \hat{x}. \tag{4.7}$$

Hence for all $t \geqslant 0$,

$$p_t{}^*[(\min_i \hat{x}^i)\, \boldsymbol{\omega}] \leqslant p_t{}^* \hat{x} \leqslant p_0{}^* \hat{x}. \tag{4.8}$$

From (4.8) we get $|p_t{}^*| \leqslant (p_0{}^*\hat{x}/\min_i \hat{x}^i) \equiv A$, say. Hence, we have

$$0 \leqslant p_t{}^* x_t{}^* \leqslant |p_t{}^*|\, |x_t{}^*| \leqslant A\, |x_t{}^*|. \tag{4.9}$$

But the right side in (4.9) goes to 0 along a subsequence in view of (4.5). Hence, $p_t{}^* x_t{}^*$ goes to 0 along a subsequence. As $\lim_{T\to\infty} p_T{}^* x_T{}^*$ exists, we must have $\lim_{T\to\infty} p_T{}^* x_T{}^* = 0$, establishing (4.2).

Turning to the demonstration of (4.5), our strategy is to arrive at a contradiction by supposing that (4.5) does *not* hold. If (4.5) does *not* hold, there is some $\alpha' > 0$ and some $T' > 0$ such that

$$|x_t{}^*| > \alpha' \qquad \text{for all } t \geqslant T'. \tag{4.10}$$

There are three main steps leading to a contradiction from (4.10). We take them in the following order:

Step 1. We have to show that (4.10) implies that

$$\lim_{t\to\infty} d[(x_t^*, y_{t+1}^*); F^*] = 0. \tag{4.11}$$

Step 2. By using Proposition 4.3 we have to prove that (4.10) and (4.11) imply that there is some $\mathbf{e} = e\boldsymbol{\omega} \gg 0$ and some $T_0 \geqslant T'$ such that

$$x_t^* \geqslant \mathbf{e} \gg 0 \qquad \text{for all } t \geqslant T_0. \tag{4.12}$$

Step 3. Using (4.12) and (4.10), construct a feasible program $(\tilde{x}, \tilde{y}, \tilde{c})$ from $\mathbf{x} \gg 0$ such that $\tilde{c}_t \geqslant c_t^*$ for all t and $\tilde{c}_t > c_t^*$ for at least one t. This means that (x^*, y^*, c^*) is not efficient, a contradiction that establishes (4.5).

To fill in the details of Step 1, suppose that (4.11) is false. This means that for some $\epsilon > 0$

$$d[(x_t^*, y_{t+1}^*); F^*] \geqslant \epsilon \qquad \text{for an infinite number of periods.} \tag{4.13}$$

Among the first T periods, let $N(T)$ be the number of periods in which (4.13) holds. One has, from (3.1) (recall that $\hat{\lambda} = 1$),

$$\sum_{t=1}^{T} \hat{p}c_t^* \leqslant \hat{p}\mathbf{x} + \sum_{t=1}^{T} [\hat{p}y_t^* - \hat{p}x_{t-1}^*] - \hat{p}x_T^* \tag{4.14}$$

and $\hat{p}y_t^* \leqslant \hat{p}x_{t-1}^*$ for *all* t, since $(x_{t-1}^*, y_t^*) \in \mathscr{T}$. Now, for each of the $N(T)$ periods in which (4.13) is supposed to hold, Proposition 4.1 can be applied to get

$$0 \leqslant \sum_{t=1}^{T} \hat{p}c_t^* \leqslant \hat{p}\mathbf{x} - N(T)\,\delta(\hat{p}x_{t-1}^*) - \hat{p}x_T^*. \tag{4.15}$$

As $\hat{p} \gg 0$, and for $t \geqslant T'$, $| x_t^* | > \alpha'$, by (4.10), if $N(T)$ goes to infinity with T, the right-hand side is negative for a sufficiently large $N(T)$, whereas the left side is always nonnegative, a contradiction establishing (4.11).

Coming to Step 2, observe that if $(x_t^*, y_{t+1}^*) \in F^*$ for some $t \geqslant T'$, then by following the argument used in Proposition 4.3 (with α replaced by α' in (4.4)), we get some $e_{\alpha'}\boldsymbol{\omega} \gg 0$ such that $x_t^* \geqslant e_{\alpha'}\boldsymbol{\omega}$ for any such t. To extablish (4.12), therefore, one can just as well assume that (x_t^*, y_{t+1}^*) does *not* belong to F^* for any t. Choose $\epsilon > 0$ such that $\epsilon < e_\alpha/2$ where $e_\alpha > 0$ is given by Proposition 4.3. Given this $\epsilon > 0$, according to (4.11), there is some $T_0 > T'$ such that for each $t \geqslant T_0$, there is $(x_t, y_{t+1}) \in F^*$ such that

$$d[(x_t^*, y_{t+1}^*); (x_t, y_{t+1})] < \epsilon. \tag{4.16}$$

Since F^* is a cone, we can take $|(x_t, y_{j+1})| = 1$ without loss of generality (recalling the definition (2.1) of the angular distance used above). Now, $(x_t, y_{t+1}) \in F^*$ and $|(x_t, y_{t+1})| = 1$ implies by Proposition 4.3,

$$x_t \geqslant e_\alpha \boldsymbol{\omega} \gg 0 \qquad \text{for all } t \geqslant T_0. \tag{4.17}$$

But according to (4.15) and the fact that $|(x_t^*, y_{t+1}^*)| \geqslant | x_t^* | > \alpha'$,

$$\text{for} \quad t \geqslant T_0, \qquad | x_t^{*i}/|(x_t^*, y_{t+1}^*)| - x_t^i | < \epsilon \qquad \text{for all } i;$$

or,

$$\text{for} \quad t \geqslant T_0, \qquad x_t^{*i} > |(x_t^*, y_{t+1}^*)| \, (x_t^i - \epsilon) \qquad \text{for all } i; \tag{4.18}$$

or,

$$\text{for} \quad t \geqslant T_0, \qquad x_t^{*i} > \alpha'(e^c - \epsilon) \equiv e > 0 \qquad \text{for all } i.$$

This completes the proof of (4.12).

For constructing a program $(\tilde{x}, \tilde{y}, \tilde{c})$ that would contradict the efficiency of (x^*, y^*, c^*), note that $\hat{x} \leqslant (\max_i \hat{x}^i)\, \boldsymbol{\omega}$ implies that $\boldsymbol{\omega} \geqslant \hat{x}/\max_i \hat{x}^i$. Hence for all $t \geqslant T_0$,

$$x_t^* \geqslant e\boldsymbol{\omega} \geqslant e[\hat{x}/\max_i \hat{x}^i] \geqslant \mathbf{m}\hat{x}, \tag{4.19}$$

where $\mathbf{m} = e/\max_i \hat{x}^i$. Since $\hat{p} \gg 0$, and from (4.13), $\sum_{t=1}^{\infty} \hat{p}c_t^* \leqslant \hat{p}\mathbf{x}$, we have $\sum_{t=1}^{\infty} | c_t^* | \leqslant M$ where M is a (finite) positive number. It follows that there is $t' \geqslant T_0$ such that

$$\sum_{t=t'+1}^{\infty} | c_t^* |/\min_i \hat{x}^i \leqslant \mathbf{m}/2. \tag{4.20}$$

Setting $\theta_t^* = | c_t^* |/\min_i \hat{x}^i$, we can rewrite (4.19) as

$$\sum_{t=t'+1}^{\infty} \theta_t^* \leqslant \mathbf{m}/2. \tag{4.21}$$

We now construct a program $\langle \hat{x}, \tilde{y}, \tilde{c} \rangle$ from $\tilde{x} \gg 0$ as follows:

(a) $\tilde{x}_t = x_t^*, \tilde{y}_t^* = y_t^*, \tilde{c}_t = c_t^*$ for $t = 1, \ldots, t' - 1$,

(b) $\tilde{y}_{t'} = y_{t'}, \tilde{x}_{t'} = (\mathbf{m}/2)\, \hat{x}, \tilde{c}_{t'} = \tilde{y}_{t'} - \tilde{x}_{t'}$,

(c) $\tilde{y}_t = \tilde{x}_{t-1}, \tilde{x}_t = [\mathbf{m}/2 - \sum_{s=t'+1}^{t} \theta_s^*]\, \hat{x}, \tilde{c}_t = \tilde{y}_t - \tilde{x}_t$ for $t > t'$.

In order to check that the program is feasible we proceed as follows:

(1) Nonnegativity: Obvious for $t \leqslant t' - 1$.

For $t = t'$, note that $\tilde{c}_{t'} = \tilde{y}_{t'} - \tilde{x}_{t'} = y_{t'}^* - (\mathbf{m}/2)\,\hat{x} = (c_{t'}^* + x_{t'}^*) - (\mathbf{m}/2)\,\hat{x} \geqslant (c_{t'}^* + \mathbf{m}\hat{x}) - (\mathbf{m}/2)\,\hat{x} \gg c_{t'}^* \geqslant 0$. Also, it is clear that $x_{t'}^* \gg 0$ and $y_{t'}^* \gg 0$. For $t > t'$, $\tilde{x}_t \gg 0$, since $\sum_{s=t'+1}^{t} \theta_s^* \leqslant \mathbf{m}/2$ (by 4.21)) and $\tilde{c}_t = \tilde{y}_t - \tilde{x}_t = \tilde{x}_{t-1} - \tilde{x}_t = \theta_t^*\hat{x} \gg 0$.

(2) $(\tilde{x}_t, \tilde{y}_{t+1}) \in \mathscr{T}$, for all $t \geqslant 0$, since this is obvious for $t \leqslant t' - 1$, and for $t \geqslant t'$, $(\tilde{x}_t, \tilde{y}_{t-1}) \in \mathscr{T}$ as $\tilde{x}_t$ is proportional to $\hat{x}$, and $\mathscr{T}$ is a cone containing $(\hat{x}, \hat{x})$.

(3) Obviously, $\tilde{y}_t = \tilde{x}_t + \tilde{c}_t$. Thus we see that $(\tilde{x}, \tilde{y}, \tilde{c})$ is feasible from $\mathbf{x}$. Finally, $\tilde{c}_t = c_t^*$ for $t = 1,..., t' - 1$; $\tilde{c}_{t'} \gg c_{t'}$ (as verified in (1) above); for $t > t'$, $\tilde{c}_t = \tilde{y}_t - \tilde{x}_t = \tilde{x}_{t-1} - \tilde{x}_t = \theta_j^*\hat{x} = [\| c_t^* \|/\min_i \hat{x}^i]\,\hat{x} \geqslant [\| c_t^* \|/\min_i \hat{x}^i][\min_i \hat{x}^i\omega] = \| c_t^* \|\,\omega \geqslant c_t^*$.

Hence (x^*, y^*, c^*) is inefficient, completing the proof of Theorem 4.1.

Q.E.D.

Remark. It is instructive to look at an example of an efficient program that satisfies the transversality condition at one system of competitive prices, while violating it at another. Note that in this example, (A.5) is not satisfied. $\mathscr{T} = \{(x, y): Bz \geqslant y,\ Az \leqslant x \text{ for some } z \geqslant 0\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 1 \end{bmatrix}.$$

Here $\hat{x} = [{}^{\frac{1}{2}}_{1}]$ $(= \hat{z})$ is a von Neumann stock vector, and $\hat{p} = [0, 1]$ a von Neumann price vector. Let $\mathbf{x} = [{}^{1}_{1}]$ be the initial stock vector. Define $x_t^* = [{}^{0}_{1}]$, for all $t \geqslant 1$, $y_1^* = [{}^{\frac{3}{4}}_{1}]$ and $y_{t+1}^* = [{}^{\frac{1}{4}}_{1}]$ for all $t \geqslant 1$, $c_1^* = [{}^{\frac{3}{4}}_{0}]$ and $c_{t+1}^* = [{}^{\frac{1}{4}}_{0}]$ for all $t \geqslant 1$. This program is efficient. Two competitive price systems are (a) $p_t^* = \hat{p}$ for all $t \geqslant 0$, and (b) $q_0^* = (\frac{1}{2}, \frac{1}{4})$, $q_1^* = (1, 0)$, $q_t^* = (0, 0)$ for all $t \geqslant 2$. Clearly the transversality condition does *not* hold for the price system (a), while it does for the price system (b). Since much has been aid about characterizing efficiency in terms of present value maximization, note that the efficient program of this example actually *minimizes* the value of consumption over the set of all feasible consumption sequences at the price system (a).

References

1. M. Majumdar, T. Mitra, and D. McFadden, Efficiency and Pareto optimality of competitive programs in closed multisector models, *J. Econ. Theory* **13** (1976), 26–46.

2. L. McKenzie, Turnpike theorems for a generalized Leontief model, *Econometrica* **31** (1963), 165–180.
3. H. Nikaido, "Convex Structures and Economic Theory," Academic Press, New York, 1968.
4. R. Radner, Paths of economic growth that are optimal only with regard to final states: A turnpike theorem, *Rev. Econ. Stud.* **28** (1961), 98–104.